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1979 J. Phys. A: Math. Gen. 12 49

(http://iopscience.iop.org/0305-4470/12/1/015)

J. Phys. A: Math. Gen., Vol. 12, No. 1, 1979. Printed in Great Britain

Green function for an extended, uniformly charged nucleus

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Received 23 May 1978

Abstract. An exact, closed solution for the Coulomb Green function of an extended, uniformly charged nucleus is presented.

1. Introduction

In calculations of radiative electron capture (Martin and Glauber 1958), radiative muon capture (Rood *et al* 1974), and radiative pion capture as well as in calculations of electromagnetic corrections to the induced pseudoscalar term in ordinary muon capture (Fulcher and Mukhopadhyay 1973, 1975, Mukhopadhyay 1977), the Green function for the Coulomb field is needed to determine the capture rate. If the nuclear charge distribution is taken to be of a Saxon–Woods type, only numerical solutions can be generated for both the wavefunction and the Green function. In the case of a uniform charge distribution, we have shown that an exact closed solution for the wavefunction exists (Yano and Yano 1972). In this paper, we show a corresponding solution also exists for the Green function, a fact that does not seem to be generally recognised (see, for example, Fulcher and Mukhopadhyay 1973, 1975). This solution is of some practical use since the leading correction to the physical finite size effect is given by the uniform charge distribution (see, for example, Fujii *et al* 1968).

In § 2, we write down the equation for the Green function and consider solutions of the corresponding homogeneous solutions. In an appendix, we give expressions for the connection coefficients.

2. Homogeneous solutions

The differential equation which defines the Green function for a uniform charge distribution is

$$[-(\hbar^2/2m)\nabla^2+V(r)-E]G_{\mathrm{E}}(\boldsymbol{r},\boldsymbol{r}')=-\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{2.1}$$

where

$$V(r)=\begin{cases}-(Ze^2/2R_{\mathrm{c}})(3-r^2/R_{\mathrm{c}}^2), & 0\leqslant r\leqslant R_{\mathrm{c}},\\ -Ze^2/r, & r\geqslant R_{\mathrm{c}}.\end{cases} \tag{2.2}$$

Performing a partial wave expansion, $G_E(\boldsymbol{r}, \boldsymbol{r}')$ may be written as

$$G_E(\boldsymbol{r}, \boldsymbol{r}') = \sum_{l,m} C_l K_l(r_<) T_l(r_>) Y_{lm}(\hat{r}) Y^*_{lm}(\hat{r}'). \tag{2.3}$$

K_l and T_l are solutions of the homogeneous radial equation and are the objects to be determined. To this end, we must separately consider inside and outside solutions of the homogeneous radial equation.

2.1. *Inside solutions*

The inside ($r < R_c$) homogeneous radial equation is

$$\left(\frac{1}{r^2}\right)\frac{d}{dr}\left(r^2\frac{dR_{in}}{dr}\right) - \left(\frac{l(l+1)}{r^2} + \mu^2 - \frac{3mZe^2}{\hbar^2 R_c} + \frac{mZe^2 r^2}{\hbar^2 R_c^3}\right) = R_{in} = 0 \tag{2.4}$$

where

$$\mu^2 = -2mE/\hbar^2. \tag{2.5}$$

In equation (2.5), μ^2 is positive (negative) if E is negative (positive). We discuss the negative energy case as this is the situation found in the processes mentioned above. An obvious modification is needed for the positive energy case.

If the modified radial function χ_{in} is introduced by

$$\chi_{in} = \rho R_{in}, \tag{2.6}$$

$$\rho = \lambda^{1/2} r, \tag{2.7}$$

$$\lambda^2 = mZe^2/\hbar^2 R_c^3, \tag{2.8}$$

$$\gamma = (-\mu^2 + 3mZe^2/\hbar^2 R_c)/2\lambda, \tag{2.9}$$

then equation (2.4) becomes

$$d^2\chi_{in}/d\rho^2 + [2\gamma - \rho^2 - l(l+1)/\rho^2]\chi_{in} = 0. \tag{2.10}$$

The further substitution

$$z = \rho^2 \tag{2.11}$$

$$\chi_{in} = z^{(l+1)/2} \exp(-z/2) L(z), \tag{2.12}$$

converts equation (2.10) to

$$z\, d^2L/dz^2 + (l + \tfrac{3}{2}) - z)\, dL/dz - \tfrac{1}{4}(2l + 3 - 2\gamma)L = 0 \tag{2.13}$$

which is Kummer's equation (Abramowitz and Stegun 1965).

The two independent solutions of equation (2.13) are $M(a, b, z)$ and $U(a, b, z)$ with

$$a = (2l + 3 - 2\gamma)/4, \tag{2.14}$$

$$b = l + \tfrac{3}{2}. \tag{2.15}$$

Therefore, the two corresponding solutions of the inside radial equations are

$$R_{in}^{(1)} = (\lambda^{1/2} r)^l \exp(-\tfrac{1}{2}\lambda r^2) M(a, b, \lambda r^2), \tag{2.16}$$

$$R_{in}^{(2)} = (\lambda^{1/2} r)^l \exp(-\tfrac{1}{2}\lambda r^2) U(a, b, \lambda r^2). \tag{2.17}$$

2.2. Outside solutions

For $r > R_c$, the homogeneous equation is that for a point charge. The solutions are standard and are given by the Whittaker functions. It is, however, more convenient for our purposes to express the solutions in terms of Kummer functions. Explicitly,

$$R^{(1)}_{\text{out}} = (2\mu r)^l \exp(-\mu r) M(c, d, 2\mu r), \tag{2.18}$$

$$R^{(2)}_{\text{out}} = (2\mu r)^l \exp(-\mu r) U(c, d, 2\mu r), \tag{2.19}$$

where

$$c = l + 1 - \eta, \tag{2.20}$$

$$d = 2l + 2, \tag{2.21}$$

$$\eta = mZe^2/\hbar^2\mu. \tag{2.22}$$

3. The Green function

From their definition in equation (2.3), $K_l(r_<)$ must be finite at the origin and $T_l(r_>)$ must vanish at infinity. We need the corresponding behaviour of M and U. For the inside solutions, as $r \to 0$,

$$M(a, b, \lambda r^2) \to 1, \tag{3.1}$$

$$U(a, b, \lambda r^2) \to \Gamma(l + \tfrac{1}{2})(\lambda r^2)^{-l-\frac{1}{2}}/\Gamma(a). \tag{3.2}$$

It follows that $K_l(r_<)$ is given by equation (2.16) when $r_<$ is less than R_c.

For the outside solutions, as $r \to \infty$,

$$M(c, d, 2\mu r) \to \Gamma(d) \exp(2\mu r)(2\mu r)^{-l-1-\eta}/\Gamma(c) \tag{3.3}$$

$$U(c, d, 2\mu r) \to (2\mu r)^{-l-1+\eta}. \tag{3.4}$$

It follows that $T_l(r_>)$ is given by equation (2.19) for $r_> > R_c$.

The expressions for $K_l(r_<)$ and $T_l(r_>)$ when $r_< > R_c$ and $r_> < R_c$ respectively can be written as follows:

$$K_l(r_<) = AR^{(1)}_{\text{out}}(r_<) + BR^{(2)}_{\text{out}}(r_<) \qquad r_< > R_c \tag{3.5}$$

$$T_l(r_>) = CR^{(1)}_{\text{in}}(r_>) + DR^{(2)}_{\text{in}}(r_>) \qquad r_> < R_c. \tag{3.6}$$

The coefficients A, B, C, D are determined from the condition of continuity at $r = r' = R_c$. They are ratios of Wronskians evaluated at R_c and their derivation is given in the appendix.

We can now state the results for K_l and T_l:

$$K_l(r) = \begin{cases} (\lambda^{1/2} r)^l \exp(-\tfrac{1}{2}\lambda r^2) M(a, b, \lambda r^2), & r \leqslant R_c \quad (3.7) \\ (2\mu r)^l \exp(-\mu r)(AM(c, d, 2\mu r) + BU(c, d, 2\mu r)), & r \geqslant R_c \quad (3.8) \end{cases}$$

$$T_l(r) = \begin{cases} (\lambda^{1/2} r)^l \exp(-\tfrac{1}{2}\lambda r^2)(CM(a, b, \lambda r^2) + DU(a, b, \lambda r^2)), & r \leqslant R_c \quad (3.9) \\ (2\mu r)^l \exp(-\mu r) U(c, d, 2\mu r), & r \geqslant R_c \quad (3.10) \end{cases}$$

with A, B, C, D given by equations (A.12)–(A.15).

The coefficient C_l of equation (2.3) can be obtained in the usual way by integrating over the δ function discontinuity. The result is

$$C_l = 2\mu\Gamma(l+1-\eta)/(A\Gamma(2l+2)). \tag{3.11}$$

Finally, we show that when $R_c \to 0$, the Green function reduces to that for the point charge. For this case, we need to use only the part of $K_l(r)$ and $T_l(r)$ with $r > R_c$, in equations (3.8), (3.10). Therefore, we consider only the coefficients A and B given by equations (A.12), (A.13). A simple calculation shows that

$$A \to R_c^{(-3/4)l} \tag{3.12}$$

$$B \to R_c^{2+(5/4)l} \tag{3.13}$$

as $R_c \to 0$.

The fact that A diverges as $R_c \to 0$ is not a problem as the A which appears in K_l is cancelled by the A in the denominator of C_l, equation (3.11). The Green function therefore becomes

$$\begin{aligned}\mathscr{G}_l(r, r'; E) &= C_l K_l(r_<) T_l(r_>) \\ &\to 2\mu[\Gamma(c)/\Gamma(d)] \exp(-\mu r_<)(2\mu r_<)^l M(c, d, 2\mu r_<) \\ &\quad \times \exp(-\mu r_>)(2\mu r_>)^l U(c, d, 2\mu r_>)\end{aligned} \tag{3.14}$$

which is exactly the Green function for a point charge.

Acknowledgments

It is a pleasure to thank H P C Rood for useful discussions. One of us (AFY) also acknowledges support from a CSULB Foundation Summer grant. This work was supported in part by a Research Corporation grant.

Appendix. Connection coefficients

The connection coefficients in equations (3.8), (3.9) are

$$A = [W(R_{\text{in}}^{(1)}, R_{\text{out}}^{(2)})/W(R_{\text{out}}^{(2)}, R_{\text{out}}^{(2)})]_{R_c} \tag{A.1}$$

$$B = [W(R_{\text{out}}^{(1)}, R_{\text{in}}^{(1)})/W_{\text{out}}]_{R_c} \tag{A.2}$$

$$C = [W(R_{\text{out}}^{(2)}, R_{\text{in}}^{(2)})/W_{\text{in}}]_{R_c} \tag{A.3}$$

$$D = [W(R_{\text{in}}^{(1)}, R_{\text{out}}^{(2)})/W_{\text{in}}]_{R_c} \tag{A.4}$$

where the Wronskian of the outside solutions is

$$W_{\text{out}} = W(R_{\text{out}}^{(1)}, R_{\text{out}}^{(2)}) = -\Gamma(2l+2)(2\mu r^2\Gamma(l+1-\eta))^{-1}. \tag{A.5}$$

The other four Wronskians are

$$W_{\text{in}} = W(R_{\text{in}}^{(1)}, R_{\text{in}}^{(2)}) = -2\Gamma(l+\tfrac{3}{2})(\lambda^{1/2}r^2\Gamma(\tfrac{1}{2}l+\tfrac{3}{4}-\tfrac{1}{2}\gamma))^{-1} \tag{A.6}$$

$$W(R_{\text{in}}^{(1)}, R_{\text{out}}^{(2)})$$
$$= \rho^l \exp(-\tfrac{1}{2}\rho^2) \exp(y/2) y^l \{M(a, b, \rho^2) U(c, d, y)(-\mu + \lambda r) + M(a, b, \rho^2) \times U(c+1, d+1, y)(-2\mu c) + M(a+1, b+1, \rho^2) U(c, d, y)(-\lambda r) \times [1 - 2\gamma(2l+3)^{-1}]\} \tag{A.7}$$

$$W(R_{\text{out}}^{(1)}, R_{\text{in}}^{(1)})$$
$$= y^l \exp(-y/2) \rho^l \exp(\tfrac{1}{2}\rho^2) \{M(c, d, y) M(a, b, \rho^2)(-\lambda r + \mu) + M(c, d, y) M(a+1, b+1, \rho^2) \lambda r [1 - 2\gamma/(2l+3)] + M(c+1, d+1, y) M(a, b, \rho^2)(-\mu)[1 - \eta/(l+1)]\} \tag{A.8}$$

$$W(R_{\text{out}}^{(2)}, R_{\text{in}}^{(2)})$$
$$= y^l \exp(-y/2) \rho^l \exp(-\tfrac{1}{2}\rho^2) [U(c, d, y) U(a, b, \rho^2)(-\lambda r + \mu) + U(c, d, y) U(a+1, b+1, \rho^2)(-\lambda r)(b - \gamma) + U(c+1, d+1, y) U(a, b, \rho^2) 2\mu (l + 1 - \eta)]. \tag{A.9}$$

In the preceding calculations, we have used the equations

$$\frac{\mathrm{d}M(a, b, z)}{\mathrm{d}z} = (a/b) M(a+1, b+1, z) \tag{A.10}$$

$$\frac{\mathrm{d}U(a, b, z)}{\mathrm{d}z} = -a U(a+1, b+1, z). \tag{A.11}$$

The connection coefficients are explicitly,

$$A = -2\mu R_c^2 (\Gamma(c)/\Gamma(d)) \rho_c^l \exp(-\tfrac{1}{2}\rho_c^2) \exp(y_c/2)(y_c)^l \times \{M(a, b, \rho_c^2) U(c, d, y_c)(-\mu + \lambda R_c) + M(a, b, \rho_c^2) U(c+1, d+1, y_c) \times (-2\mu c) + M(a+1, b+1, \rho_c^2) U(c, d, y_c)(-\lambda R_c)[1 - 2\gamma/(2l+3)]\} \tag{A.12}$$

$$B = -2\mu R_c^2 (\Gamma(c)/\Gamma(d)) \exp(-y_c/2) y_c^l \rho_c^l \exp(-\tfrac{1}{2}\rho_c^2) \{M(c, d, y_c) M(a, b, \rho_c^2)(-\lambda R_c + \mu) + M(c, d, y_c) M(a+1, b+1, \rho_c^2) \lambda R_c [1 - 2\gamma/(2l+3)] + M(c+1, d+1, y_c) M(a, b, \rho_c^2)(-\mu)[1 - \eta/(l+1)]\} \tag{A.13}$$

$$C = -\lambda^{1/2} R_c^2 (\Gamma(a)/2\Gamma(b)) \exp(-y_c/2) y_c^l \rho_c^l \exp(-\tfrac{1}{2}\rho_c^2) \times [U(c, d, y_c) U(a, b, \rho_c^2)(-\lambda R_c + \mu) + U(c, d, y_c) U(a+1, b+1, \rho_c^2)(-\lambda R_c)(b - \gamma) + U(c+1, d+1, y_c) U(a, b, \rho_c^2) 2\mu c] \tag{A.14}$$

$$D = -\lambda^{1/2} R_c^2 (\Gamma(a)/2\Gamma(b)) \rho_c^l \exp(-\tfrac{1}{2}\rho_c^2) \exp(-y_c/2) y_c^l \times \{M(a, b, \rho_c^2) U(c, d, y_c)(-\mu + \lambda R_c) + M(a, b, \rho_c^2) U(c+1, d+1, y_c)(-2\mu c) + M(a+1, b+1, \rho_c^2) U(c, d, y_c)(-\lambda R_c)[1 - 2\gamma/(2l+3)]\} \tag{A.15}$$

where

$$\rho_c = \lambda^{1/2} R_c \tag{A.16}$$

$$y_c = 2\mu R_c. \tag{A.17}$$

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